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PROCEEDINGS
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A STUDY OF FEATHER CHARACTER IN LIMBS TRANSPLANTED BETWEEN EMBRYOS OF DIFFERENT BIRD SPECIES

By Herbert L. Eastlick

Department of Zoölogy, University of Missouri[1]

Communicated October 14, 1939

This paper reports some observations made upon pigment and feathers in transplanted limbs of avian embryos. The transplants include all possible reciprocal combinations between guinea fowl, duck, turkey and chick embryos and, in addition, grafts of chukar partridge and quail limb buds to embryonic chick hosts. For a preliminary report of this work see Eastlick.[2]

Three- to four-day embryos were used as donors and hosts. An opening 5–7 mm. square was cut in the shell directly over the host embryo, the vitelline membrane slit and the amnion torn open whenever necessary. A limb bud was severed from a donor embryo, transferred to the lateral body wall of the host and either placed into the intra-embryonic coelom (Hamburger[3]) or attached to the right lateral body wall of the host (Eastlick[4]).

Transplants placed into the intra-embryonic coelom usually interfere with the retraction of the yolk sac so that the host dies before hatching. However, in one case a host to which a duck leg had been transplanted was autopsied 26 days after hatching and a graft 5 cm. in length was found within the coelom. Transplants to the lateral body wall interfere much less frequently with the development of the host so that many are able to hatch and survive. The oldest host of this type, a chicken bearing an extra duck leg, was 15 weeks old on October 12. So far, 48 chick hosts bearing extra limbs from various avian embryos have hatched; this represents approximately 17% of the total number of embryos with flank grafts allowed to reach the age of hatching.

Chick to Chick Transplants.—In these experiments White Silkie Bantam embryos were used as donors while the hosts were White Leghorn, New Hampshire Red and Barred Plymouth Rock embryos. Silkie feathers, with the possible exception of the proximal portions of the major wing and tail feathers, are distinctive since they have a plume-like or "hairy" appearance due to the absence of hooklets (barbicels) on the barbules. Of the 115

transplants made, 62 were successful and 8 hatched. Two of the transplanted legs possess Silkie type feathers since barbicels are lacking at 8 weeks of age, while the other leg has feathers which are intermediate between donor and host types in that some of the barbules have barbicels while others do not. All of the transplanted wings possess feathers of the intermediate type with the distal ends of the feathers usually presenting a frayed appearance while the proximal ends are of host type. The frayed condition is brought about by the barbules tending to separate because of the absence of barbicels or the lack of a number sufficient to hold them together. Frequently two or more adjacent barbules have numerous hooklets and resemble the host feather type while other barbules in the immediate vicinity are bare. It is suggested, tentatively, that this result is brought about by the mesoderm of the host invading the graft and progressively altering the development of the donor follicles in some unknown manner. More experiments will be made in an attempt to gain further information on this point.

It is well known that Leghorns are fast feathering, that Barred Rocks feather slowly while Silkies seem to be intermediate. The data secured in this study suggest that the feathers of a transplanted limb maintain their inherent (donor) growth rate rather than assuming that of the host. For example, a Brown Leghorn wing bud transplanted to a Barred Plymouth Rock produced feathers which developed more rapidly than the plumage of the host (Fig. 1). Moreover, the feathers of Silkie wings developed on White Leghorn hosts did not grow as fast as the wing feathers of the host but a Silkie leg grafted to a Barred Rock produced feathers which grew more rapidly than those on the host's own legs.

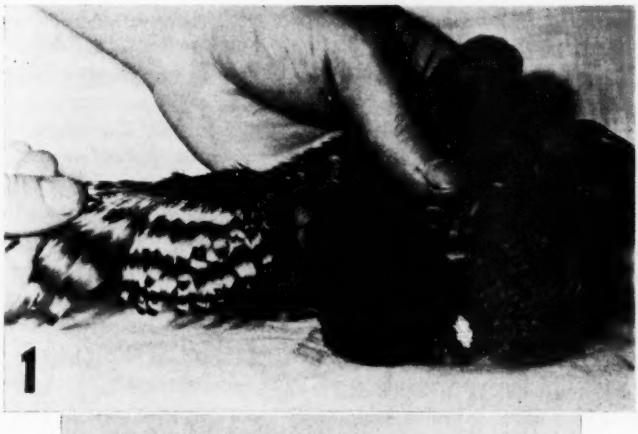
These results differ from those obtained by Willier and Rawles[5] who made skin grafts of White Silkie Bantams to other breeds and found that the transplanted Silkie skin developed feathers which possessed the form, arrangement and growth rate of the host. Willier and Rawles transplanted a restricted piece of ectoderm plus small amounts of adhering mesoderm, while in the present study the entire limb bud with its ectoderm and mesoderm was used. It is suggested that the amount of mesoderm which is included with the donor tissue may account for the divergent results which have been secured in the two laboratories.

When a comparison is made between grafts of White Leghorn and Silkie limb buds, the latter may be distinguished by their smaller size at comparable stages and the presence of feathers along the outer portions of the shank and foot in more advanced stages.

Two chickens have hatched in which Silkie limbs were transplanted to New Hampshire Red hosts. In both of these the melanophores of the host invaded the transplant and pigmented the feathers.

When a posterior limb bud of a 60-hour Silkie Bantam was removed by

making the cuts at the level of the posterior cardinal vein and transplanted to a White Leghorn host, the resulting graft lacked dermal pigment on the shank and possessed colorless skin due to the absence of subcutaneous chromatophores. It is suggested that these two types of pigment cells



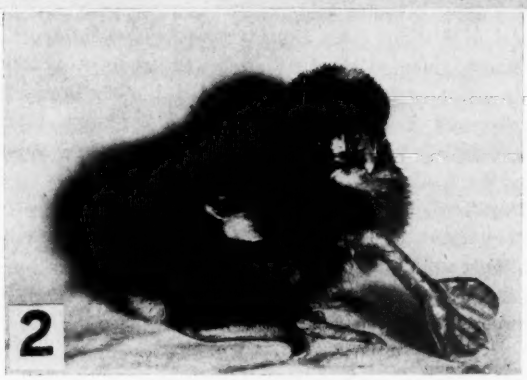


FIGURE 1

Twenty-one day Barred Plymouth Rock host whose right wing is from a Brown Leghorn donor. The feathers on the graft have grown more rapidly than those on the host's wing (left).

FIGURE 2

Nine-day Barred Plymouth Rock host bearing a White Pekin duck leg. Melanophores of the host have pigmented the down on the transplant.

are derived from the neural crest since all grafts which included any of the crest were pigmented.

No antagonistic reactions have been observed between donor and host

in these transplantations which is in marked contrast to the intense inflammation and destruction of the epidermis of duck, guinea hen, chukar partridge and quail limb buds on chick hosts.

Reciprocal Transplants between Chick and Duck Embryos.—Of the 132 operations made by grafting Mallard and White Pekin duck limb buds to White Leghorn and Barred Plymouth Rock hosts, 62 were successful and 15 hatched. Typical web-footed duck legs developed upon the chick hosts. The extra duck legs persist after the hatching of the host and, for a time, grow more rapidly that the host's own legs.

Transplants of White Pekin limb buds to the flanks of Barred Rock hosts invariably develop pigmented down. The melanophores of the chick migrate into the extra leg in large numbers as is shown by the fact that the skin of the shank, toes and web of the graft as well as the feathers became pigmented (Fig. 2). White Pekin transplants made to White Leghorn hosts always develop non-pigmented feathers. Mallard limb buds transplanted to White Leghorns develop pigmented down when neural crest cells are included with the donor tissue but non-colored down when the crest cells are excluded. Similar results have been obtained by Eastlick[4] in transplants of limb buds between pigmented and non-pigmented chick embryos.

Mallard melanophores are able to migrate extensively in White Leghorn hosts for in a number of cases the pigment cells wandered out from the transplant and pigmented the feathers on the right wing and the major portion of the down on the right lateral body wall of the host. In such cases the feathers beyond the limit of the transplant are unquestionably of host origin and are merely pigmented by the donor melanophores. The down was replaced by feathers having the same form, distribution and rate of growth as host feathers on the unoperated side.

The down on the extra limb, which probably is of donor origin, is lost following an intense inflammation in the epidermis of the graft. This reaction does not proceed beyond the limits of the duck skin as may be noted by the difference in color of the skins of the donor and host. Later feathers which appear to be of host origin develop and cover the graft. A more extensive account of the tissue reaction and replacement phenomena will be reported later.

Of the 60 operations in which chick limb buds were transplanted to duck hosts, 20 cases were successful but none survived beyond the time of hatching. White Leghorn grafts attached to the mesenteries of Mallard hosts remain non-colored while transplants made to the lateral body wall always possess pigmented down. In the former, melanophores of the host were unable to migrate into the graft while they succeeded in doing so in the latter cases. At present it is not clear why melanophores are unable to migrate across the mesenteries. Transplants of Barred Rock limb buds to White Pekin ducks were colored or non-colored depending upon the pres-

ence or absence of neural crest cells. Barred Rock melanophores are able to migrate extensively since the wing and body wall of a number of hosts became pigmented when the transplant included neural crest cells.

Sixteen reciprocal transplants have been made between Mallard and White Pekin embryos in which 11 cases were positive but none hatched. The White Pekin transplants remained non-colored when attached to the mesenteries but became pigmented when attached to the lateral body wall of the hosts. Mallard transplants which contained neural crest cells sometimes pigmented large areas of the Pekin hosts but when neural crest cells were absent from the donor tissue, non-colored grafts resulted.

Reciprocal Transplants between Guinea Fowl and Chick Embryos.—Of the 96 guinea fowl to chick transplants which were made, 49 were successful and 9 hatched. Pearl guinea fowl embryos were used as donors in all cases and most of the hosts were White Leghorns although a few were Barred Rocks. It was noted that the majority of grafts possessed non-colored feathers even when the protocols stated that neural crest cells should have been included with the donor limb buds. In an effort to insure that crest cells were included with the transplant, the donor buds were severed by making the cuts adjacent to the neural tube. Even in such instances the grafts were covered with only partially pigmented down and the melanophores failed to migrate into the adjacent tissues of the host. It seemed that this reaction might be due to (1) the failure of neural crest cells of the guinea to migrate laterally from the cord as rapidly as is the case in chick or duck embryos or (2) an antagonism between chick and guinea tissues. In an attempt to gain evidence on this point a half-portion of the neural tube and the neural crest between two or more somites were transplanted to the flank of White Leghorn hosts. Rather small pigmented areas resulted although similar transplants of Barred Rock tissues produced extensive areas of pigmented down. Moreover, an extra guinea fowl limb on a chick host fails to survive more than a few days after hatching, while an extra duck limb persists and grows. These results suggest that chick hosts afford an unfavorable environment to the cells and tissues of the guinea fowl. An explanation of the factors underlying the inability of the guinea tissue to survive awaits further study.

Transplants of guinea fowl limb buds to the flank of Barred Rock hosts invariably bear pigmented down even when donor neural crest cells are excluded. This is due to the migration of chick melanophores into the graft. The down is not replaced by juvenile feathers since the graft undergoes necrosis soon after the time of hatching.

Forty chick to guinea fowl transplants have been made of which 20 were positive but none hatched. White Leghorn buds grafted to the flank of guinea fowl hosts became fully pigmented when a broad attachment existed between the host and transplant but in case of a narrow bridge, the feathers

on the graft usually became colored only in part. All grafts connected with the mesenteries were non-pigmented.

Reciprocal Transplants between Chick and Turkey Embryos.—Of the 122 chick to turkey grafts which have been made, 60 were positive and one embryo hatched but lived only a day. White Leghorn limb buds transplanted to the coelom remained non-pigmented when attached to the mesenteries. Grafts within the coelom, however, developed colored down if attached to the muscles of the inner body wall. Melanophores are unable to migrate across intact parietal peritoneum or mesenteries but can cross a muscular bridge between donor and host so that donor limbs within the coelom or attached to the lateral body wall of the host become darkly pigmented. The melanophores not only pigment the down of the graft but the skin of the shank and foot as well.

Eighty-four turkey to chick transplants were made, 51 being successful. Two of the latter hatched but lived only a day. Usually turkey limbs do not develop a normal amount of muscle when transplanted to chick hosts but the skeletal elements grow quite rapidly. Some of the grafts are larger than any of the chick to chick transplants. The turkey transplants possess pigmented down only when neural crest cells are included with the donor tissue. Turkey melanophores were found to migrate rapidly into chick tissues.

Fifty-four turkey to turkey transplants have been made of which 15 were positive but none hatched. Both donors and hosts were of the Bronze variety. Transplants which became attached to the mesenteries developed colored plumage only when neural crest cells were included while all grafts on the outer body wall of the hosts were colored.

Reciprocal Transplants between Turkey, Duck and Guinea Fowl Embryos.—Successful grafts have been made in all possible combinations but none of the embryos hatched and no facts at variance with the preceding have been noted.

Transplants of Quail and Chukar Partridge Limb Buds to Chick Embryos.—Fifty-three transplants of quail limb buds to chick hosts have been made; 26 were successful and 7 hatched. Of the total of 63 chukar to chick grafts made primarily to White Leghorn hosts 27 were successful and of these 6 hatched. The quail and chukar partridge limb buds developed normally but the grafts are much smaller than chick to chick transplants.

Non-pigmented grafts were secured in both types by excluding the crest cells from the donor tissue. The feathers on the transplants grew for a few days but degenerated and gradually sloughed off during the period of inflammation and destruction of the epidermis which occurred usually between 8 and 14 days after hatching. After varying periods of time, the grafts ceased growing and were resorbed gradually. In two cases a few feathers, seemingly of host origin, appeared upon the graft during the

period of resorption. While the original feathers on these transplants possessed typical chukar partridge pigmentation and markings, the regenerated feathers were non-colored.

In several instances melanophores spread from the transplant pigmenting a considerable area of the host down which later was replaced by feathers possessing, entirely or in part, the markings of the donor but which grew at the host rate. In many cases the tips of the definitive feathers were more heavily pigmented than the bases; frequently the terminal ends were darkly colored, the mid-portions were mosaics and the bases non-colored. When such a feather was pulled the one replacing it was non-colored and regenerated at the same rate as a similar feather on the unoperated side. These feathers which seemed to be of host origin, but possessing donor type of pigmentation, were not influenced during the period when the epidermis of the transplant was destroyed.

[1] The experimental work was done at the University of Missouri where the project was aided by a grant from the University Research Council.

[2] Eastlick, H. L., *Nature*, **144**, 380–381 (1939).

[3] Hamburger, V., *Jour. Exptl. Zoöl.*, **77**, 379–399 (1938).

[4] Eastlick, H. L., *Ibid.*, **82**, 131–157 (1939).

[5] Willier, B. H., and M. E. Rawles, *Proc. Nat. Acad. Sci.*, **24**, 446–452 (1938).

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*THE BREEDING OF SOME RABBITS PRODUCED BY RECIPIENTS OF ARTIFICIALLY ACTIVATED OVA**

BY GREGORY PINCUS

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Communicated October 4, 1939

In a previous paper[1] I reported that three litters of young were obtained from rabbit does which had received transplantations of eggs previously subjected to artificially activating treatments. One was a litter of fifteen, of which three were males and twelve females. Two were litters of one, each female. Since the latter were produced by does made pseudopregnant by pituitary extract there could be no doubt that the young arose by parthenogenetic development. This was confirmed by the fact that the coat color of the offspring was that to be expected from the transplanted ova. In the case of the first litter, however, a probability exists that some of the offspring might have been developed from fertilized eggs since the recipient female was made pseudopregnant by mating with a vasectomized male. If this male were imperfectly vasectomized, or retained sperm for some time after vasectomy, he might have been the parent of the young.

The ova transplanted into the first female were superovulated by pitui-

tary extract and treated as follows: (1) 28 (from an agouti donor) heated at 47°C. for 3 minutes and transplanted into the left fallopian tube of the recipient; (2) 23 (from a chinchilla donor) placed in 1.9% NaCl for 7 minutes and transplanted to the right fallopian tube of the recipient. The latter was an agouti female. The vasectomized male was a black, non-agouti and non-chinchilla (e.g., *aaCC*). Of the fifteen young one was a chinchilla agouti female like the chinchilla donor, and therefore must have come from a parthenogenetic ovum. Of the remaining young 7 were agouti (*AC*) and 7 were non-agouti (*aa*). These might have been derived from eggs produced by the recipient and fertilized or from the fertilization of the transplanted ova. Palpation before birth revealed young in both uterine horns, but the exact number in each could not be determined.

Three of these fourteen young have survived to breeding age. One is an agouti male, two are non-agouti females. The male has been mated to females of four types with the results indicated below:

FEMALE GENOTYPE		OFFSPRING PHENOTYPE		
		AC	Ac^{ch}	*aC* or ac^{ch} †
(Chinchilla)	$AAc^{ch}c^{ch}$	12	25	..
(Albino)	*AAcc*	5	5	4
(Black)	*aaCC*	2	..	1
(Agouti)	$AACc^{ch}$	2	2	..

† Non-agouti chinchilla cannot with certainty be distinguished from non-agouti non-chinchilla.

Of these 58 offspring 36 were females, 22 males. It is obvious that the genotype of the male parent is $AaCc^{ch}$. He might have arisen from one of the transplanted chinchilla ova fertilized by a *Y*-chromosome bearing sperm of the vasectomized male or from any other Ac^{ch} egg so fertilized (conceivably the host female or the agouti donor might have been heterozygous for chinchilla). The excess of chinchilla young among his offspring is not quite significantly different from the chance expectation, but is interesting, and further matings are being instituted to see if it continues.

Of the two non-agouti females one is definitely chinchilla. She has produced 18 offspring in three litters by a chinchilla male. All these young were chinchilla. She must therefore be parthenogenetic. The second has produced 30 young in five litters, 17 females and 13 males. The matings were all to a chinchilla male ($AAc^{ch}c^{ch}$). All the offspring were *AC* phenotypes. Therefore the mother is *aaCC*, and might have been derived from an *aC* egg fertilized by an *aC* sperm. It is notable that this female has consistently produced runts (1 to 3 per litter), which are born dead or die shortly after birth.

Not previously reported is a litter of two chinchilla young produced by

an albino recipient made pseudopregnant by pituitary extract injection. This female had received 18 ova, taken 19 hours after an ovulating injection, from a superovulated doe whose genotype was $AAc^{ch}c$. The ova were immersed in hypotonic (60%) Ringer-Locke solution for 4 minutes, transferred to rabbit serum for 15 minutes, then immersed in hypertonic Ringer-Locke solution for 4 to 6 minutes, transferred again to serum and then transplanted to the left fallopian tube of the recipient.

Of the two females obtained one has thus far proved sterile. The other has produced one litter of 9 young after a mating to an albino male. Six of the nine young were chinchilla, three were albino. The segregation of albino young indicates that the mother is $c^{ch}c$. Since tubal ova in the rabbit have already undergone the first maturation division these data indicate that the first meiosis is not necessarily reductional. It is hoped that further production of parthenogenetic rabbits will give further information on this and related questions. Methods for increased production are being investigated.

These data together with those previously reported indicate that viable parthenogenetic rabbits can be obtained from artificially activated eggs and that fertile females can be produced.

* Aided by grants from the American Academy of Arts and Sciences and the Rockefeller Foundation.

[1] Pincus, G., *Jour. Exptl. Zoöl.*, **82**, 89 (1939).

ON PERIODICITIES IN MEASURES OF THE SOLAR CONSTANT

BY THEODORE EUGENE STERNE

HARVARD COLLEGE OBSERVATORY

Communicated October 11, 1939

1. *Introduction.*—If the solar "constant" changes, then a positive correlation is to be expected between measures that have been made nearly simultaneously at different stations. Yet the existence of a positive correlation between such measures, even if the correlation is statistically significant, does not by itself show that the solar constant has changed. If measures at the different stations contain changing systematic errors, then any positive correlation that there may be, between those errors, must tend to produce a positive correlation between the measures. Miss Paranjpe has discussed[1] the statistical significance of the correlations between measures of the solar constant made at different stations of the Astrophysical Observatory of the Smithsonian Institution. She finds that some of the correlations are statistically significant while others are not. She attributes

the statistically significant ones to correlations, some arising from volcanic dust and others from the methods employed in the numerical reductions, between the systematic errors.

The physical significance of the correlations is controversial, and I do not wish to join the controversy* here. There is, however, another method of attack upon the problem of determining whether or not the solar constant changes. That other method is to look for periodicities in the measured values of the solar constant. If the solar constant could be shown to contain one or more periodic components, it would thereby be shown to change. Dr. C. G. Abbot suggests,[2] as the result of harmonic analysis, that measures of the solar constant contain periodic components, not necessarily sinusoidal, having periods of 7, 8, $9^3/_4$, 11, 21, 25, 34, $39^1/_2$, 46, 68 and 92 months, and 23 years. Since Abbot does not test the suggested periodicities for statistical significance, the question naturally arises whether some or all of these periodicities may not be merely statistical fluctuations in the random errors of the measures. In this communication, the preceding question will be answered, in so far as it relates to the 540 values of the solar constant (mean values taken over consecutive intervals of $^1/_{36}$ of a year) that are listed by Abbot on page 12 of reference 2. The values cover the interval from the beginning of 1920 to the end of 1934.

Before proceeding to tests of significance, it is well perhaps to mention here an objection[1] of Dr. Paranjpe's to the periodicities suggested by Abbot. It has been shown by Brunt[3] that if a series of observations, whose standard deviation is σ, is the resultant of a number of periodic terms and of a remainder whose standard deviation is σ_r, then $\sigma^2 = \sigma_r^2 + {}^1/_2\Sigma R^2$ where ΣR^2 is the sum of the squares of the amplitudes of the periodic terms in question.† Since σ_r^2 cannot be negative, ${}^1/_2\Sigma R^2$ clearly cannot exceed σ^2. Using for the R's the "amplitudes" assigned by Abbot to his suggested periods, Paranjpe finds that ${}^1/_2\Sigma R^2$ exceeds σ^2 by a factor of nearly four, and she therefore concludes that the amplitudes are impossible. The word "amplitude," however, as used by Abbot, refers not to R but to $2R$. It refers to the full range of variation from maximum to minimum, whereas R refers to the coefficient of a sine-term and is thus one-half of the full range. The quantities listed by Paranjpe as values of ${}^1/_2\Sigma R^2$ are really values of $2\Sigma R^2$ and should be divided by four; when this is done the discrepancies disappear. Her objection seems to rest upon a misunderstanding, and to be invalid.

2. *The Least Squares Solutions.*—The observational material that has been analyzed consists of the 540 decade-values that have been previously mentioned, a "decade" being $^1/_{36}$ of a year. It is convenient to number the decades by integers, N, that run from zero through 539; decade zero being the first in January, 1920. Corresponding to any period, P decades in length, a phase φ can be assigned to each decade-value by the relation $\varphi =$

$360°(N/P) - \alpha$, where α is an arbitrarily disposable constant whose value does not significantly influence the probability P_{bc}, later to be introduced as a criterion of significance. With each decade-value used as the second member, M, of an observational equation of the form

$$a + b \sin \varphi + c \cos \varphi = M, \qquad (1)$$

the values of the constants a, b and c, as well as the mean (root mean square) errors of these constants, can then be found by least squares. In this procedure each least squares solution would contain 540 observational equations. It is better to reduce the number of observational equations by grouping the decade-values into phase-classes, and for all the periods in the accompanying table, except the first, 25 phase-classes were used. The first phase-class then included all decade-values with phases between 0° and 14°.4, the second all phases between 14°.4 and 28°.8, etc., and the last all phases between 345°.6 and 360°. Each class included its upper limit. For the first period, 21 phase-classes were used, with centers at 0°, (360/21)°, (720/21)°, etc. The mean of the decade-values in each class was then used as the second member of an observational equation of the form (1), whose phase corresponded to the middle of the class. The observational equations, equal in number to the number of phase-classes, were assigned equal weights[‡] in the least squares solutions.

The results of the solutions are given in the table. The consecutive columns contain the period in months; the value of P (as close as possible to the value given by Abbot) actually used in computing the phases; the value of α that was employed; the value of b, with its mean error found in the usual way from the residuals of the observational equations; the value of c with its mean error; the value of the amplitude A, equal to $(b^2 + c^2)^{1/2}$; and the value of P_{bc}. The values of a are all nearly equal to 1.9409 calories per square centimeter per minute, are of little interest, and are not given; A, b and c are expressed in thousandths of a calorie per square centimeter per minute.

If t_b and t_c are the ratios of b and c to their mean errors (t_b and t_c are distributed nearly normally by chance, with unit standard deviation), then[4] the probability, P_{bc}, that fluctuations of sampling[§] will give rise to a value of $t_b^2 + t_c^2$ equal to or larger than the value observed, is given by $\exp[-\frac{1}{2}(t_b^2 + t_c^2)]$. A small value of P_{bc} suggests, with a degree of assurance depending upon its smallness, that the periodicity cannot reasonably be attributed to statistical fluctuations in the random errors of the material examined, and therefore suggests that the periodicity is really present in the decade-values. It is seen that the periods of 7, 8 and 34 months, suggested by Abbot, have reasonably large P_{bc}'s. They can reasonably be attributed to statistical fluctuations, and there is no need to assign any statistical or physical significance to them. The seven other periods sug-

gested by Abbot, in the table, have such very small P_{bc}'s that statistical significance is strongly suggested for them.

PERIOD (MONTHS)	P	α (DEGREES)	b	c	A	P_{bc}
7	21	120/7	0.48 ± 0.39	-0.04 ± 0.39	0.48	0.47
8	24.099	0	-0.03 ± 0.37	-0.89 ± 0.37	0.89	0.055
$9^3/_4$	29.25	0	2.13 ± 0.40	0.35 ± 0.40	2.16	5×10^{-7}
11	32.704	0	3.11 ± 0.34	0.11 ± 0.34	3.11	6×10^{-19}
21	63	0	-0.30 ± 0.50	2.26 ± 0.50	2.28	3×10^{-5}
25	75	$-12/5$	-3.59 ± 0.49	1.65 ± 0.49	3.95	7×10^{-15}
34	102	0	-0.32 ± 0.92	-0.65 ± 0.92	0.72	0.73
$39^1/_2$	118.5	0	3.59 ± 0.82	-5.24 ± 0.82	6.35	9×10^{-14}
46	138	0	3.18 ± 0.77	2.46 ± 0.77	4.02	1×10^{-6}
68	204	0	1.69 ± 0.90	4.78 ± 0.90	5.07	1×10^{-7}

A point that must be borne in mind is that the periods suggested by Abbot were selected by him after a harmonic analysis of the material. The number of independent trial periods that could have been proposed, had no selection been made, must be taken into account. This number, which depends upon the resolving power of a periodogram of the Schuster type, is rather small. If the duration of the series under investigation is T, then the resolving power** in the neighborhood of a trial period P can be shown to be very nearly P^2/T. It follows that in a series that covers 180 months there can be only about 23 independent trial periods between 7 and 68 months. With more numerous trial periods, an appreciable amplitude for any particular one would automatically entail appreciable amplitudes for the adjacent ones. The statistical significance of the seven periods with small P_{bc}'s depends upon the smallness of P_7, which may be defined as the probability that out of 23 independent trial periods, seven or more should by chance have P_{bc}'s that are smaller than 3×10^{-5}. From the Binomial law, P_7 is found to be 5×10^{-27}, and this value is so small that the seven periods, 9.75, 11, 21, 25, $39^1/_2$,[††] 46[††] and 68 months must be considered to have a very high collective significance.

It is well perhaps to state here that the conclusions that have been reached by the least squares analysis already described, about the significance of particular periods suggested by Abbot, have been confirmed in all cases by an indepedent procedure[‡‡] that employed the analysis of variance, and in which the variance within phase-classes was compared with the variance between them.

3. *Interpretation of the Results.*—It has been shown that seven periodicities mentioned in the preceding section cannot reasonably be ascribed to statistical fluctuations in the random errors of the observational material. With a very high degree of assurance, one must consider them to be really present in the values of the solar constant that have been analyzed. It does not necessarily follow that they must be present in the solar constant itself.

The values that have been analyzed may be considered to be composed of the solar constant, plus random errors, plus possible systematic errors. All that one can say with assurance is that the periodicities in question must be present either in the solar constant, in the systematic errors or in both. It is difficult, but not impossible, to imagine how periodicities could occur in the systematic errors. The reader will have noticed that tests of the statistical significance of Abbot's suggested periods of 92 months and 23 years have not been included in the table. Only two periods of 92 months, and not even one period of 23 years, can be included in the interval of 180 months covered by the observations. For the 23-year period, the presence of almost any gradually changing systematic error at all would result in non-vanishing values for b and c, which could easily be statistically significant. For the period of 92 months, some similarity between the gradual variations of the systematic error during the first and second repetitions is by no means improbable, and significant values of b and c could thus be obtained. In other words, harmonic analysis of a gradually changing quantity is readily able to yield appreciable values of the b's and c's of trial periods that are too long to repeat very often during the observational series. It is so likely to yield them for periods that repeat only once that we have not included the two longest periods, suggested by Abbot, in our analysis; and although it is considerably less likely to yield them for the shorter periods it can, nevertheless, yield them. It is thus possible for gradually changing systematic errors, which give rise to significant values of b and c for certain trial periods, to contain those periods in a "real," and yet perhaps not in a physically significant sense.

Besides the periods in the table, a period of one year, not suggested by Abbot, has also been examined. An annual period is the most plausible period to expect, on physical grounds, in the systematic errors. It could result from an annual term in atmospheric effects that have not been completely eliminated from the observations. Twelve phase-classes were used (the centers being at 0°, 30°, 60°, etc.), with α equal to 20°, and b and c were found to be 0.65 and 1.32 with mean errors of 0.25. The value of P_{bc} is 3×10^{-8}, so that a real annual periodicity is strongly suggested. It lies, probably, in the systematic errors.

It is a pleasure to acknowledge the assistance that has been given to me in the computations by A. L. Jennings, J. Levine, R. Loevinger, D. A. MacRae, F. S. Patterson, P. J. Rubinstein and G. D. Sterne.

4. *Summary.*—Five hundred and forty ten-day mean values of the solar constant, published by Abbot in *Smithsonian Miscellaneous Collections*, **94**, No. 10, p. 12 (1935), have been examined by least squares. The coefficients, and their mean errors, have been determined of the sine and cosine terms corresponding to each of ten periods, lying between 7 and 68 months, that have been suggested by Abbot. It is found that seven periods (9.75, 11,

21, 25, $39^1/_2$, 46 and 68 months) are of strongly suggested statistical significance and must in all probability be present either in the systematic errors of the ten-day means, in the solar radiation, or in both. Three periods can reasonably be attributed to statistical fluctuations. An annual periodicity not suggested by Abbot is found; it is of strongly suggested statistical significance, and it is thought likely that it occurs in the systematic errors.

* For Abbot's reply to Dr. Paranjpe, see reference 7.

† The relation follows at once from the consideration that each periodic term of amplitude R contributes $^1/_2R^2$ to the total variance, σ^2, of the series.

‡ The most accurate weights would be the populations of the phase-classes. Within a particular solution, such populations do not differ very much, and thus the weights would not differ very much. Weights were so assigned at first for some of the periods, and the results do not differ significantly from those in the table; thus the eighth period gives, with accurate weights, $b = 3.51 \pm 0.82$, $c = -5.06 \pm 0.80$, $P_{bc} = 2 \times 10^{-13}$. Least squares values are generally insensitive to the choice of weights. Errors of even a million-fold would be inconsequential in P_{bc}'s as small as 10^{-13}, in which absolute, not necessarily relative, accuracy is needed.

§ In reference 4, b and c are slightly correlated; here they are completely independent because of the adoption of equal weights. Since the errors of the b's and c's are linear functions of the errors of the M's, the former errors are very nearly normally distributed even if the latter errors are not. See Eddington, reference 5. The mean errors in the table are estimates, obtained from 18 degrees of freedom in the first line and from 22 in the others; these numbers are large enough to render nearly normal the chance distributions of t_b and t_c. The values of P_{bc} given in the table are of course only approximations, but they are amply accurate enough to decide questions about statistical significance. Probabilities smaller than 0.01 are usually considered to suggest statistical significance strongly; here the "leeway" is enormous.

** The half-width of a periodogram-peak.

†† These two periods do not appear to be well resolved, and may be related.

‡‡ For a description of the use of the analysis of variance, in tests of the significance of periodicities, the reader is referred to the author's paper on the rotation of Uranus, reference 6.

[1] Paranjpe, *Jour. Roy. Meteorolog. Soc.*, **64,** 459 (1938).
[2] Abbot, *Smithsonian Misc. Coll.*, **94,** No. 10 (1935).
[3] Brunt, *Combination of Observations*.
[4] Sterne, *Ap. Jour.*, **81,** 45 (1935).
[5] Eddington, *Proc. Phys. Soc.*, **45,** 271 (1933).
[6] Sterne, *Bull. Harvard Coll. Obs.*, No. 904, 27 (1936).
[7] Abbot, *Jour. Roy. Meteorolog. Soc.*, **65,** 215 (1939).

GALACTIC AND EXTRAGALACTIC STUDIES, II. NOTES ON THE PECULIAR STELLAR SYSTEMS IN SCULPTOR AND FORNAX

BY HARLOW SHAPLEY

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Communicated October 7, 1939

1. *Results of a General Search.*—The examination of one hundred and fifty-two small-scale plates, covering something more than fifteen thousand square degrees of the sky in galactic latitudes higher than $\pm 20°$, has failed to show any other objects similar to the stellar systems in Sculptor and Fornax reported last year.[1] The plates were made at the Boyden Station, Bloemfontein, with the AX camera attached to the Bruce telescope; the exposures are each uniformly three hours in length; the scale is 1 mm = 11'.5. Since the new objects in Sculptor and Fornax are readily shown on such plates, we must infer from the fruitless search that stellar systems of this sort are infrequent in the neighborhood of our Galaxy and probably not frequent within a million light years.[2]

In the course of the search, two new globular clusters and two new objects of the Magellanic Cloud type have been noted. The new globular clusters are in (or superposed upon) the Fornax system. One, of magnitude 14.2, is in the position $2^h\,35^m\,58^s$, $-34°58'$ (1900), practically at the center of the system.[3] The other, a loose globular cluster near the southern edge, is in the position $2^h\,34^m\,33^s$, $-35°14'$, and of magnitude approximately 14.4. There is a third very faint cluster of unidentified character, magnitude 16.6:, at $2^h\,35^m\,56'$, $-34°51'$, which is also probably a part of the Fornax system.

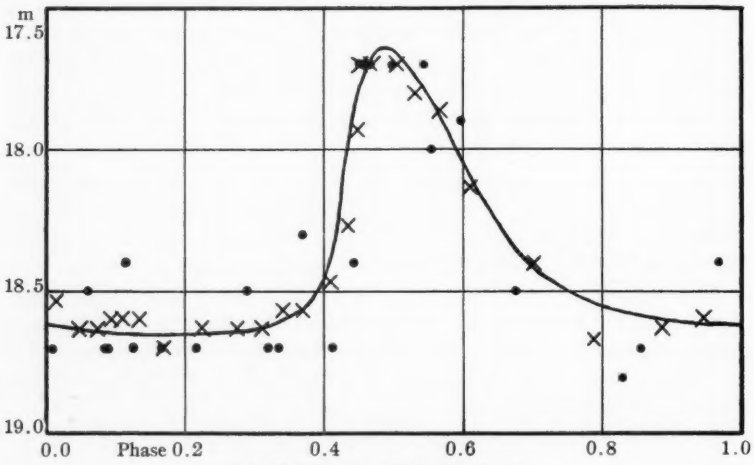
2. *Description of the Two Objects.*—Because of the probable significance of the Sculptor and Fornax systems in the sequence of galactic forms, a further report is now made, with special reference to the distances of the two objects and to the brightness and distribution of stars throughout the one in Fornax.

An important addition to our knowledge of the new systems has resulted from the preliminary investigations by Baade and Hubble with the aid of photographs made with the 100-inch reflector at Mount Wilson.[4] Chiefly through the discovery of faint variable stars in the Sculptor system, the Mount Wilson observers show that the preferred hypothesis as to distance, dimensions and luminosity is almost certainly correct; that is, the systems are of the size of the average galaxy, but have low total intrinsic brightness and are devoid of supergiant stars. They are members of the local group of galaxies.

The positions, apparent magnitudes and angular dimensions of the two objects are as follows:

	SCULPTOR CLUSTER	FORNAX CLUSTER
Right ascension (1900)	$0^h\ 55.^m4$	$2^h\ 35.^m6$
Declination (1900)	$-34°\ 14'$	$-34°\ 53'$
Galactic latitude	$-83°$	$-64°$
Galactic longitude	243°	203°
Total photographic magnitude	9.0:	9.0:
Magnitude of brightest stars	17.8	19.3
Mean angular diameter	75′	65′

3. *Classical Cepheids in the Sculptor System.*—Plates made with the 24-inch and 60-inch telescopes in South Africa permit the derivation of provisional periods and light curves for the two bright variable stars in the Sculptor cluster, which were found on Mount Wilson plates. As suspected, the stars apparently are classical Cepheids. Provisionally, periods of about 6.518 days and 1.35 days, for Variables A and B, respectively, have been determined by Miss Swope from the few observations now available, but for the second star other values are not excluded.



Variable "A" in Sculptor System.

In the accompanying figure the dots indicate single observations on Variable A; the crosses are running means. The magnitude scale is necessarily precarious for such faint objects in declination $-34°$. With no correction for space absorption the median photographic magnitudes of the two variables are 18.2 and 18.1; and the mean absolute magnitude, with half weight for Variable B, is -1.2.

4. *Distances and Absolute Magnitudes.*—The high galactic latitude, $\beta = -83°$, and the faintness of the classical Cepheids in the Sculptor cluster make it extremely improbable that the variables are merely superposed members of the Galaxy. Accepting that they are members of the system, we find, in the mean, 76 kiloparsecs for the distance of the Sculptor system, in agreement with my originally suggested value of 80 kiloparsecs, based on the magnitudes of the brighter stars, and with the value of 84 kiloparsecs estimated by Baade and Hubble from the magnitudes at maxima of the numerous faint (presumably cluster-type) variables that they observed. There are, however, serious uncertainties involved and the probable error of the deduced distance may easily exceed ten per cent.

The Sculptor and Fornax systems, though 21° apart, appear together on some of the small-scale plates now available at Harvard and the total apparent magnitudes can therefore be compared directly. Revising somewhat an earlier estimate, we now find that the Fornax system is at least as bright as the Sculptor cluster; both are uncertainly placed at apparent magnitude 9.0. The absolute magnitudes, on the other hand, appear to differ considerably. The plates made with the Boyden Station 60-inch reflector show, as also do the Mount Wilson plates, that the most luminous individual stars in the Fornax system are about a magnitude and a half fainter than those in the Sculptor cluster. We infer, therefore, that either the relative frequencies of the absolute magnitudes of their stars are quite different, or the Fornax cluster is twice as far away as the Sculptor cluster and intrinsically brighter. Baade and Hubble suggest, on the basis of the magnitudes of the brighter stars in the presumably associated globular cluster NGC 1049, that the Fornax cluster is 2.2 times as distant as the system in Sculptor.

With the distances as indicated above, the total absolute magnitudes of the Sculptor and Fornax systems, -10.5: and -12:, are of the same order as the absolute magnitudes of the brightest globular clusters and the small unresolved spheroidal galaxies, such as the companions of the Andromeda Nebula (M 32 and NGC 205). Similar also in absolute magnitude, Baade and Hubble point out, is IC 1613, a dwarf irregular galaxy (Magellanic type) which is being extensively studied by Baade.

The similarity in total luminosity and probably in total mass of three such systems as Omega Centauri, NGC 205, and the Sculptor cluster, should be of significance in the interpretation of large stellar systems, notwithstanding the differences of these organizations in structure. All three are centrally, but not highly, concentrated and appear to be oblately spheroidal (NGC 205 particularly oblate). All are devoid of supergiant stars. NGC 205 is intermediate in dimensions, the Sculptor cluster in luminosity.

It is interesting to note that a view of our galaxy from the region of the

Andromeda Nebula would show it to be accompanied by several "small spheroidal galaxies"—that is, by the brightest of the globular clusters. Omega Centauri, 47 Tucanae, and similar giant globular clusters should probably be given galactic rank if the companions to the Andromeda Nebula are placed in that category. The Sculptor and Fornax systems might be called greatly expanded giant clusters; but speculation on their nature and relationship to other galactic structures could better await improved information about the luminosity functions involved and the total magnitudes.

5. *Star Counts in the Fornax System.*—A survey of the distribution of several thousand of the brightest stars in the Sculptor cluster has been given in an earlier paper.[5] The individual stars in the Fornax cluster are too faint for satisfactory counting on the Bruce long-exposure photographs, since the brightest individual stars are fainter than magnitude 19. An attempt to count the stars and study the distribution on reflector plates gives only provisional results because the small curved field of the reflector is unsuited to the study of such an extended object. On a plate showing stars on the optical axis to magnitude 20.5, approximately, 4800 stars have been counted in an area of thirty-five per cent of a square degree, and in a central area of eight per cent of a square degree, where the distance correction is unimportant, 2300 stars, of which approximately six per cent are members of the foreground. On the basis of these counts, and the distribution of stars on a series of surrounding long-exposure reflector plates, we conclude that the structure of the Fornax cluster is much the same as that of the nearer system in Sculptor.

6. *Angular and Linear Dimensions.*—The star counts fail to give the full extent of the Fornax system, but a measure of its dimensions has been derived from twenty tracings, made with the photoelectric densitometer, on two three-hour plates of the AX series. The means of the measures in four position angles by Shapley and Miss Patterson are as follows:

	Core		Envelope	
E–W,	38′	± 2.′1,	71′	± 1.′9
NE–SW	44	2.6	63	1.5
N–S	41	2.7	71	1.5
NW–SE	34	2.0	52	3.1

The tabulated mean errors indicate only the uncertainty of the measurements. A three-hour Bruce plate shows a hundred faint background galaxies in the area covered by the Fornax system, the richness being practically the same as in the surrounding area and showing the general transparency of the system.

It is difficult in measuring such a faint object to determine with accuracy where the cluster fades into the galactic star field. But the diameter of the Fornax system is certainly not less than 40′, corresponding

to about two kiloparsecs, and more probably it is at least fifty per cent larger. The mean angular diameter of the Sculptor system, derived from the star counts on two Bruce plates,[5] is 75′, corresponding to a linear diameter of 1.8 kiloparsecs. These linear dimensions are comparable with those of average spheroidal galaxies and with that of the main body of the Small Magellanic Cloud.

From the tabulated measures it is seen that the inner part of the Fornax cluster is distinctly elongated, NE to SW, although the faint outermost envelope is more nearly circular. The phenomenon of a corona of stars, spheroidal in shape whatever the basic form of the underlying structure, is thus indicated in projection for both of these new objects and is beginning to appear as a significant structural feature common in stellar systems. Already it has been shown that an "escape atmosphere" surrounds the Andromeda Nebula. The galactic system apparently is similarly constructed, as illustrated by the observations reported in the preceding paper of this series. Around the two Magellanic Clouds are also such extensions—envelopes in which are found scattered giant stars, star clusters and classical Cepheid variables.

[1] The objects were described in detail at the Cambridge meeting of the British Association for the Advancement of Science, August, 1938; see also *Harv. Bull.* **908** (1938) and *Nature*, **143**, 715 (1938).

[2] Examination of the AX plates has been carried through chiefly by Miss Martha Dowse and Miss Rebecca Jones, who have had extensive experience in the study of faint galaxies.

[3] This object is also noted by Baade and Hubble; they give 15.0 as a preliminary estimate of the apparent magnitude; *Pub. Astr. Soc. Pac.*, **51** (February, 1939).

[4] Loc. cit., *supra*.

[5] *Harv. Bull.* **908** (1938).

THREE SUPERNOVAE IN THE SPIRAL NGC 3184

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In the recent discovery in the northern spiral NGC 3184 of three highly luminous variable stars, all probably of the supernova class, the observatories at Palomar, Mount Wilson and Harvard have collaborated. The three stars appeared within an interval of sixteen years, two of them indeed were at maximum in 1921—an indication that in some external systems there may be a particularly high frequency of supernovae. (The general infrequence of such outbursts has been established by Zwicky at Palomar, and verified at least qualitatively by Harvard observers.) Similarly in both NGC 4321 and NGC 6946 two supernovae have appeared, separated by in-

tervals of thirteen and twenty-two years, respectively. All three of these spirals are of class Sc and much alike in general appearance; their planes are nearly perpendicular to the line of sight, which facilitates somewhat the detection of supernovae.

The present brief account of the supernovae in NGC 3184 is made with the aid of communications from Dr. Fritz Zwicky of Palomar and Dr. E. P. Hubble of Mount Wilson.

Equatorial and galactic coördinates of the nucleus of NGC 3184 are $10^h\ 12^m.2, +41°55'$ (1900); 145°, +57°, and the total photographic magnitude is 11.8.[1] From Hubble's data[2] the distance is found to be about four million light years.

No. 1.—The supernova 149″ south of the nucleus and 5″ east, reported on Harvard Announcement Card 494, had been found on seven plates of various series made between December 9, 1937, and March 6, 1938. Later it was learned that the object had already been noted by Dr. Zwicky on a Palomar plate of early April, 1938, the magnitude then being about 16.8, three magnitudes fainter than the maximum recorded on the Harvard plates. The object does not appear on any other plates examined at Palomar, Mount Wilson, Lick or Harvard.

No. 2.—"During the examination of the old plates, however, Zwicky and Hubble found a bright star on a 100-inch plate taken by Duncan on April 6, 1921, in the position 160″ south and 32″ following the nucleus of 3184."[3] This second object does not appear on the Harvard patrol plates of 1921 which were unfortunately less sensitive that year than before or after. From a print of the Mount Wilson 100-inch plate the magnitude can be estimated at 13.9 on the basis of a comparison star sequence derived from sequences in Selected Areas 29 and 30. Harvard plates show that it must have been fainter than

magnitude	13.0	on March	29, 1921
	11.0	April	5
	12.0		12
	11.2		20

and the Mount Wilson observation of April 6, 1921, was therefore probably not far from the time of maximum brightness.

No. 3.—When seeking further images of No. 2 on Harvard plates, at the request of Dr. Zwicky, Miss R. Jones found a third supernova. With reference to the nucleus of the spiral it is 236″ south, 79″ east, not far from the position of the other two supernovae. It lies at the edge of a faint outlying spiral arm, 1500 parsecs from the nucleus if in the plane of the spiral arms. All of the positive observations obtainable from Harvard plates are given in the following tabulation, together with some significant negative observations.

OBSERVATIONS OF SUPERNOVA NO. 3

PLATE	DATE	MAGNITUDE	PLATE	DATE	MAGNITUDE
AC 23756	June 4, 1921	[12.5	AI 21438	Dec. 19, 1921	[10.8
AI 21306	Oct. 29, 1921	[11.4	AI 21452	21	11.5
AI 21372	Dec. 5, 1921	11.1	AI 21461	27	11.8
I 40780	7	11.1	AC 24549	30	[12.1
AC 24449	7	11.1	AC 24553	31	[11.5
AI 21393	7	11.1	AI 21494	31	[12.6
AI 21404	8	11.1	MC 18301	Jan. 1, 1922	12.8
AC 24462	8	11.0	AI 21502	2	[12.6
AI 21416	11	11.0	AI 21551	23	[12.8
AC 24497	19	11.2:	I 40912	26	14.1

For more than two weeks in December, 1921, supernova No. 3 equaled or exceeded in brightness the whole spiral nebula in which it appeared.

The various position coördinates in NGC 3184 were measured by Mr. Cunningham and Miss Boyd; for No. 2, in agreement with Zwicky and Hubble, they obtain the coördinates 162″ and 34″.

To test the possibility that any one of these three stars might have been bright at some time other than indicated above, we have examined 1393 Harvard photographic plates distributed throughout the interval 1890 to 1939, with all the years represented except 1895. Thirty-one per cent of these plates show stars fainter than the thirteenth magnitude; seventy-one per cent, fainter than the twelfth magnitude. No further images of the three objects were found. The negative results of this detailed search, and the position of the images in the arms of the spiral, leave little doubt but that the objects are actual members of the spiral.

[1] *Harv. Ann.*, **88**, No. 2 (1932).
[2] Mt. Wilson Contr. 548 (1936).
[3] Letter from Zwicky and Hubble, July 31, 1939.

DISTRIBUTION OF INDUCED BREAKS ALONG THE X-CHROMOSOME OF DROSOPHILA MELANOGASTER[1]

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Introduction.—A study of the distribution of induced breaks along the chromosomes of *Drosophila melanogaster* was made by Bauer, Demerec and Kaufmann.[2] From the data accumulated at that time it appeared that break frequency was proportional to length within the limits of the euchromatic portion of each salivary chromosome limb, but that the frequency

was disproportionately high in the proximal heterochromatic regions. There was a close parallel, however, for the longer autosomes between frequency of observed breaks and length as measured in the late prophase or metaphase mitotic chromosomes. Break frequency in *X*-heterochromatin was much lower than expected on this criterion (about 22 per cent, including the correction for undetectable breaks, as compared with 33.3 per cent expected).

The collection of more extensive data seemed warranted, therefore, in order to measure the reliability of these and related findings. In securing such data, which form the basis of the present report, analysis was restricted to the *X*-chromosome because of the availability of Bridges' 1938 map,[3] and the possibility presented thereby of accurately localizing break position within the limits of the lettered subdivisions. Up to the present time more than 600 additional breaks in the *X*-chromosome have been mapped from a study of about 4000 pairs of salivary glands. Final determination of break position was made in each case by the writer so that greater uniformity of cytological analysis would be guaranteed.

About 44 per cent of all breaks here reported (278 out of 627) were in chromosomes carrying the delta 49 inversion, the remainder in chromosomes having the wild-type sequence of banding. Break distribution in and adjacent to inversion points was studied as a measure of the frequency of reinversion, inasmuch as Grüneberg[4] and Emmens[5] had reported a case of reinversion correlated with a change from the phenotypic effect "roughest" to wild-type. Moreover, since the present study was initiated, there has appeared the report of Sitko[6] of increased mutability in zones adjacent to the breakage points of inverted sections.

Break Frequency Adjacent to Limits of dl-49 Inversion.—A comparison of break distribution in the *dl*-49 chromosomes with those of the wild-type reveals no significant differences. In both cases many of the same sections conform closely to expected values, and others deviate sharply therefrom. The expected values here mentioned have been calculated from the number of bands per subdivision, since this criterion was regarded as a more reliable estimate of relative length than measurements made along the map. The numbers of observed breaks in subdivisions 4*D*, *E*, *F* and in 11*E*, *F*, 12*A* (the subdivisions closest to the limits of the inversion) were 1, 2, 4 and 1, 3, 0, respectively, as compared with expected values of 1.51, 0.65, 3.02 and 2.80, 1.72, 2.16. It is apparent that there are no constellations of breaks centering about the limits of the inversion. This finding, coupled with the fact mentioned above of general similarity of break distribution in both types of chromosomes studied, points to the conclusion that the *dl*-49 sequence is neither more fragile nor more labile in its response to x-radiation than is the control. Accordingly the data as a whole may be considered and general conclusions derived therefrom.

Breaks in Proximal Heterochromatin.—The first consideration relates to the proportion of breaks which occur in the proximal heterochromatic region, subdivisions 20*A*–*F*. The value obtained, based on observed breaks, ranges between 21.6 per cent and 24.2 per cent, depending on the distribution of certain cases which are regarded as questionable because the single slide available for each rearrangement did not afford a sufficiently large number of nuclei to permit accurate diagnosis. If these questionable cases are apportioned among the chromosomes on the basis of the relative mitotic lengths of their proximal heterochromatic regions, the percentage of breaks within the *X*-heterochromatin rises to a value of about 23 per cent.

In addition to these observed breaks there are others which delimit alterations restricted to the proximal heterochromatin. Such breaks usually remain undetected, primarily because of the close pairing which obtains within the chromocentral region. Some adjustment should be made,

FIGURE 1

Intercalary regions of *X*-chromosome of *D. melanogaster* with highest break frequency. Comparison of numbers of observed and expected breaks shown below. Drawing and lettering from Bridges' 1938 map.

therefore, so as to incorporate an estimate of the frequency of such cytologically undetected changes. When this adjustment is made, using as a basis for the calculation the proportions of euchromatin to heterochromatin found in mitotic chromosomes, the percentage of breaks in *X*-heterochromatin reaches a value of 28.4 per cent. This is somewhat short of the expected 33.3 per cent but nevertheless does not exclude the possibility that break frequency is proportional to mitotic chromosome length.

Distribution of Breaks within the So-called Euchromatic Limb.—The positions of 475 breaks have been determined within the limits of the 114 subdivisions 1*A* to 19*F*. Certainly there are no long regions of the *X*-chromosome which are immune to induced breakage, since only 3 (2*D*, 6*B*, 15*B*) of the 114 subdivisions failed to show any breaks. At the other extreme, 2 subdivisions (11*A* and 12*E*) each were broken in 20 cases (Fig. 1). If break frequencies are plotted against values expected on the basis of the

number of bands per section, several striking departures from randomness are apparent, and a consideration of the entire chromosome shows that the values obtained are significantly different from expectancy (as computed by the χ^2 method, the difference equals about 7 times the standard error). The most striking differences between expected and observed frequencies were found in sections 11*A*, 12*D*, 12*E* and 19*E* (Fig. 1). Other regions with a break frequency that appears to fall outside the limits of chance distribution are 1*F*, 4*E*, 7*B*, 7*C*, 8*B*, 16*E*, 16*F* and 19*F*. It is quite probable that the accumulation of more extensive data will reveal the presence of other similar regions along the chromosome.

Discussion.—It is apparent that the findings of the present study do not support the assumption that break distribution is entirely at random along the length of the so-called euchromatic portion of the *X*-chromosome. Rather it is clear that there are subdivisions with a break frequency so high as to simulate thereby the behavior of the proximal heterochromatic region. This suggests therefore that these intercalary regions also contain heterochromatin. Certain cytological observations support this interpretation. The intercalary regions with highest break frequency, namely 11*A*, 12*D*, 12*E* and 19*E*, occasionally pair with each other, or with the chromocenter, a type of affinity that is characteristic of the heterochromatic chromocentral material. It follows, therefore, that the properties of heterochromatin may account for such pairing as well as for the high incidence of induced breakage occurring within these subdivisions. Moreover, as figure 1 shows, subdivisions 11*A*, 12*D*, 12*E* and 19*E* are characterized by repetition of sequences of bands, the so-called "reverse repeats." Conflicts in pairing between the bands of the repeat sequences of one chromosome and those of the homologous chromosome may lead to distorted or diagonal synapsis, and frequently the salivary chromosome appears to be broken completely across (occasional pairing within the repeat is found in the unsynapsed or haploid strand). Such behavior may likewise be attributable to the presence of heterochromatin. It seems unnecessary to regard these regions as "weak" spots in the sense that they represent portions of the chromosome axis in which there exists primarily a reduced longitudinal cohesion.

Some of the other regions with a break frequency that seems to fall outside the limits of chance distribution appear cytologically to represent repeats (as, for example, 8*B*); others show no conspicuous repetition of banding. There is no *a priori* reason why intercalary heterochromatic regions should be associated with repeats, except, as Muller[7] has pointed out, duplications within "inert" material would cause less functional disturbance than in "active" regions, and therefore they would have greater survival value. On the other hand, certain regions which appear cytologically to represent repeats, such as 2*B* and 3*C*, have not shown a high break frequency in the present experiments. That 2*B* and 3*C* contain

heterochromatin has been reported recently by Prokofyeva-Belgovskaya,[8] who bases her diagnosis on the cytological interpretation of pairing phenomena arising from "the capacity of inert regions to conjugate with one another." This cytological method of analysis and the break frequency approach of the present paper both lead to the conclusion that regions 7*C*, 11*A*, 12*E*, 19*E* and 20*A*–*F* contain heterochromatic material. On the other hand, several of the subdivisions which Prokofyeva interprets as heterochromatic have shown no increased break frequency in the present study, although high frequency may occur in an adjoining region. Thus, Prokofyeva reports an inert region in 17*A*, whereas in the present study this subdivision showed fewer breaks than expected (1 as compared with 5.74). Subdivision 16*F*, abutting on 17*A*, was broken, however, in 9 different cases, in comparison with the expected 3.83. One other region which may be mentioned as showing strikingly different behavior in the two studies is the tip of the *X*. Subdivisions 1*A*–1*F* all show break numbers higher than expected values, but not deviating sharply therefrom except in the case of 1*F*. Prokofyeva, however, has presented evidence that 1*A* is heterochromatic and that the presence of inert regions at the distal ends of chromosomes is a fundamental feature of their structure.[9]

In order that such differences may be appraised more critically and that pronounced trends in break frequency may be distinguished from chance deviation, the procuring of additional data seems warranted. It is apparent that in the study of Bauer, Demerec and Kaufmann,[2] divisions 11 and 12 showed the trend toward high break frequency that can now be clearly demonstrated, although with the limited number of breaks then available, the distribution for the entire chromosome gave no significant deviation from randomness. It seems best, therefore, to extend the present study before considering the apparent differences in pairing phenomena and response to irradiation shown by heterochromatin in various parts of the *X*.

The distribution of heterochromatin throughout the *X* in the course of its evolution probably has involved rearrangement primarily through inversion and the production of reverse repeats, types of changes both of which have been induced experimentally. In the *Y*-chromosome of *Drosophila miranda* as MacKnight[10] has recently reported there exists an alternating series of euchromatic and heterochromatic segments, probably arising in a similar manner.

Such evidence for the *X* suggests the possibility of the existence of similar heterochromatic regions along the limbs of the longer autosomes of *D. melanogaster*. From the standpoint of the confirmatory cytological evidence, attention is directed to the foremost of the areas of confused pairing which exist in the left limb of the second chromosome, forming the "loops" and "turn back" of Bridges' original description.[11] It seems quite probable that the production of mottling which may follow transfer of sensitive loci

to distal autosomal regions may involve juxtaposition to such intercalary heterochromatin.

Likewise, if higher break frequency occurs in certain sections of the chromosome the possibility of inversion and reinversion between two such heterochromatic regions increases. Location of original break positions in heterochromatin may account for the relative instability of some inversions as compared with the stability of those with breaks in euchromatin. In addition, the existence of intercalary heterochromatin may explain certain regional differences in distance as measured along the salivary chromosome between pairs of loci with similar crossing-over frequencies. Genes which are relatively close in the salivary chromosome may be more widely spaced in the meiotic prophases. Moreover, inert regions may have specific effects in reducing crossing-over in regions near them, probably through conflicts in pairing.

Finally some attention should be directed to the cytological implications of these findings. One of the properties of the proximal heterochromatin is its heteropyknosis in resting nuclei of the mitotic cell. If intercalary heterochromatin possesses this property, and if the frequency of breakage is a measure of the proportional length which the region concerned occupies in the mitotic chromosome, some of the intercalary heterochromatic materials should be visible as deeply staining granules in the resting or interphase nucleus. This poses the question whether some, at least, of the small spherical bodies of very early prophase chromosomes, previously described as chromomeres (cf. Kaufmann[12]), may not represent intercalary heterochromatin. Obviously the validity of such homologies can be tested only by painstaking cytogenetic analysis. Nevertheless they serve to show the various ramifications arising from the break frequency approach to the problem of the organization of the chromosome.

Summary.—The positions of 627 x-ray induced breaks have been plotted within the limits of the lettered subdivisions of the *X*-chromosome. The distribution of breaks in chromosomes carrying the *dl*-49 inversion was similar to that in chromosomes bearing the wild-type sequence of banding. About 28 per cent of breaks (including the correction for undetected breaks) occurred in the proximal heterochromatic region, 20*A*–*F*. Certain subdivisions along the so-called euchromatic portion of the chromosome, particularly 11*A*, 12*D*, 12*E* and 19*E*, showed a break frequency higher than expected on random distribution, which may be interpreted as resulting from the presence of heterochromatic material in these intercalary regions.

[1] Paper presented August 26, 1939, at Seventh International Congress of Genetics, Edinburgh.

[2] Bauer, H., Demerec, M., and Kaufmann, B. P., *Genetics*, **23**, 610–630 (1938).

[3] Bridges, C. B., *Jour. Heredity*, **29**, 11–13 (1938).

[4] Grüneberg, H., *Jour. Genetics*, **34**, 169–189 (1937).

[5] Emmens, C. W., *Ibid.*, **34**, 191–202 (1937).

[6] Sitko, N., *Memoirs on Genetics, Acad. Sci. Ukrainian S. S. R.*, **2**, 3–30; 31–49 (1938).

[7] Muller, H. J., *Collecting Net* (Woods Hole), **13**, 181, 183–195, 198 (1938).

[8] Prokofyeva-Belgovskaya, A. A., *Comp. rend. acad. sci. U. R. S. S.*, **22**, 270–273 (1939).

[9] In a more recent publication which was received subsequent to the presentation of the present article, A. A. Prokofyeva-Belgovskaya and V. V. Khvostova (*Comp. rend. acad. sci. U. R. S. S.*, **23**, 270–272, 1939) have likewise reported the use of the break frequency method as a measure of the location of inert regions. Of the total of 141 breaks reported for the entire X, 37 were in divisions 11 and 12, a proportion much higher than that of the present study. The difference may be attributable in part to the fact that the data of Prokofyeva and Khvostova are based largely on a cytogenetic analysis of lethals and not on a study of F_1 larval progeny of irradiated fathers as has been employed in the present study.

[10] MacKnight, R. H., *Genetics*, **24**, 180–201 (1939).

[11] Bridges, C. B., *Jour. Heredity*, **26**, 60–64 (1935).

[12] Kaufmann, B. P., *Jour. Morph.*, **56**, 125–155 (1934).

A THEORY OF THE COLOR OF DYES[1]

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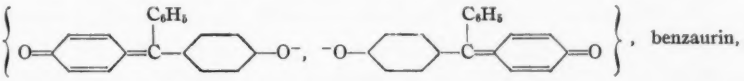
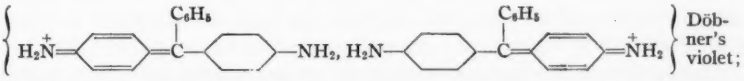
It has become recognized during recent years that the color of dyes is associated with the resonance of electric charge from atom to atom of the dye molecule.[2, 3, 4, 5, 6] Because of the complexity of the problem, however, it has not been easy to expand this idea into a theory of color permitting the rough quantitative calculation of the frequencies and intensities of the absorption bands of dyes. I have now developed a theory of this nature; the theory and some of the results of its application are described briefly in the following paragraphs.

The long-wave-length absorption band of a dye such as benzaurin or Döbner's violet has been associated with resonance of the type

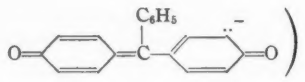




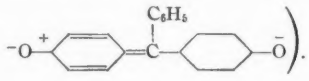
the normal state involving the symmetric combination of the structures and the upper state their antisymmetric combination. It is obvious that

many other structures are also involved, since the electrons have vanishingly small probability of jumping from one end to the other of a large molecule, but make the transit instead by moving from atom to adjacent atom.

There are two types of intermediate structures which need consideration for benzaurin, involving respectively a negatively charged carbon atom (for example, the structure



and a positively charged carbon atom (for example, the structure



All of these structures would be considered in a complete treatment; in developing a theory with minimum complexity, however, we may ignore the more unstable structures. It is seen that the structures with negatively charged carbon are less stable than those with positively charged carbon for two reasons: the latter are stabilized by Kekulé resonance of one ring (the former by none), and the degree of electronegativity of carbon favors its assumption of a positive rather than a negative charge. We accordingly neglect structures of the first type, and consider only those of the second type, corresponding to *resonance of a positive charge about the molecule* (the motion of the positive charge from atom to atom resulting, of course, from an electron jump in the opposite direction).

To set up the secular equation the following assumptions are made:

(1) All structures with a given charge distribution may be combined to a resultant structure, with energy H_{ii} given by known values of resonance energies and bond energies.

(2) The wave functions for these resultant structures are mutually orthogonal.

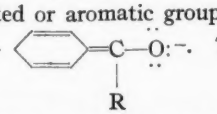
(3) The exchange integral H_{ij} between two resultant structures has the constant value ϵ if the structures differ through the motion of a charge from one carbon atom to an adjacent carbon atom (or slightly different values for C—O, C—N, etc.); for two structures differing by larger motion of the charge H_{ij} is negligibly small.

A secular equation set up in this way is similar in form to that obtained

in the molecular-orbital treatment; it differs from the latter in respect to the interpretation of the exchange integrals and the variation in H_{ii} values. Each root of our equation represents an energy level of the molecule, whereas in the molecular-orbital treatment sums of roots are taken to obtain the molecular energy levels.

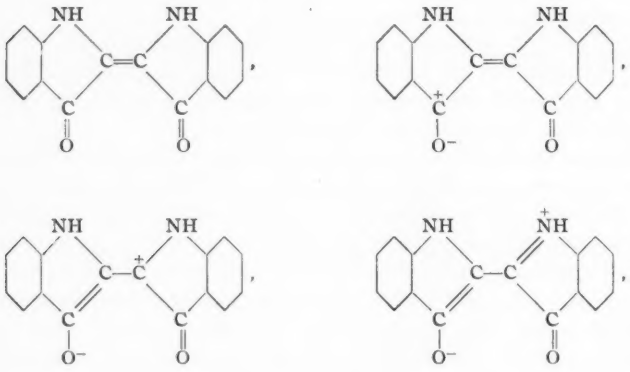
The value of the integral ϵ has been determined empirically by use of the frequency of the strong absorption band of the triphenylcarbonium cation. Using this, other energy quantities can be evaluated from the absorption frequencies for other molecules.

It is found that the Coulomb energy of a resonating charge with other charges in the molecule plays an important part in determining the energy levels and absorption frequencies. It has been recognized[7] that the absorption spectra of various substances with symmetrical molecules, such as the sulfonephthaleins R—T—R (with R = O, OH, NH_2, etc., and T = $—C_6H_5—C(C_6H_4SO_3H)—C_6H_5—$), are closely similar, whereas the related substances with unsymmetrical molecules R—T—R′ have spectra of a different type. It is found on calculation with the Coulomb energy terms ignored that the symmetrical and unsymmetrical molecules should give rise to similar spectra. The existence of two types of spectra results from the action of the Coulomb energy. In the symmetrical cation $[HO—T^+—OH]^+$ the positive charge moves freely, and in the corresponding anion $[^-O—T—O^-]^-$ the Coulomb energy remains nearly constant as the positive charge moves about in the molecule, because withdrawal from one O^- accompanies approach toward the other; hence there is close similarity in the spectra of these ions. In the unsymmetrical molecule $^-O—T^+—OH$, however, the Coulomb energy causes a pronounced change in the positions of the energy levels and the nature of the spectrum.

The carbon-oxygen double bond in all ketones involves resonance of the structures $R_2C::\ddot{O}:$ and $R_2\overset{+}{C}:\ddot{O}:^-$. For substances such as acetophenone, benzophenone, phorone, etc., in which the carbonyl bond is conjugated with an unsaturated or aromatic group, resonance with other structures occurs, such as  This deepens the color of the substance, the carbonyl absorption being shifted from the ultraviolet to the visible region of the spectrum. The bathochromic effect of the conjugation is limited by the Coulomb energy of separating the charges in the molecule; in a proton-donating solvent such as concentrated sulfuric acid the energy of separation of the charges is decreased by the field of the environment, and the color becomes deeper.

In a great many dyes, including benzaurin, discussed above, it is the

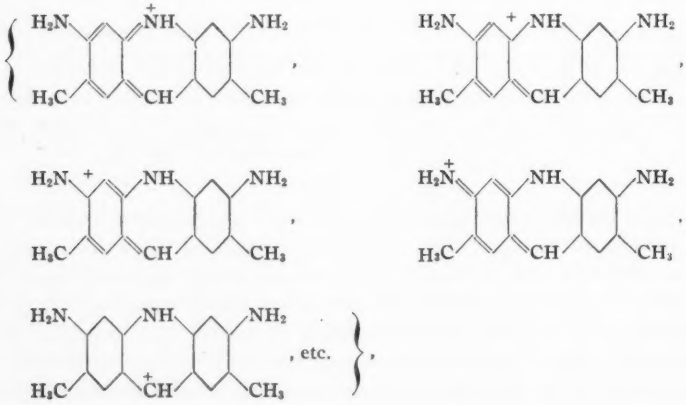
carbonyl group which is the source of the resonating charge; indigo may be mentioned as an additional example, with many structures contributing, such as



and many others, including those in which both carbonyl groups are ionized.

The $-N{=}O$, $-\overset{+}{N}\!\begin{matrix}O\\O^-\end{matrix}$ and $-N{=}N-$ groups play a similar rôle for the nitroso, nitro, and azo dyes, respectively.

The resonating positive charge in Döbner's violet is formed not by internal ionization but rather by external ionization. Other colored cations of this general type are numerous; the cation of acridine yellow, for example, can be ascribed the structure

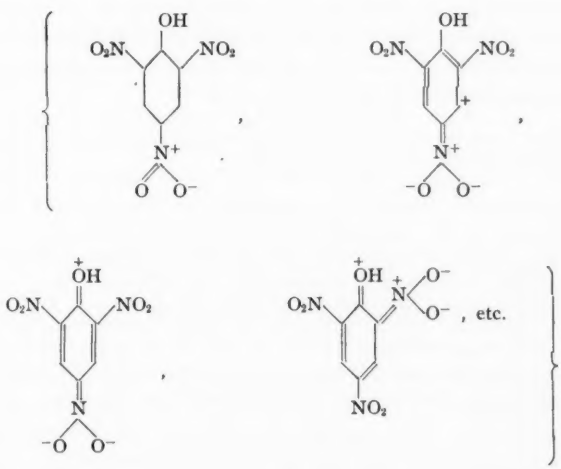


and similar structures can be written for the pyridine, quinoline, azine, oxazine, thiazine, and carbocyanine dyes, and many others

The theory permits quantitative predictions to be made regarding the intensities of absorption bands. The band of longest wave-length, representing the transition from the normal state with nodeless wave function (describing the position of the resonating charge) to the excited state with a wave function (or two wave functions, in case of degeneracy) with a single nodal plane, is predicted to be the most intense band for many dyes. (A similar result has been found for conjugated double-bond systems by Mulliken[6] by application of the molecular-orbital treatment.) The agreement between predicted and observed values of the integrated absorption coefficient is good.

The arguments given above to show that even in anions such as the benzaurin anion it is a positive charge rather than a negative charge which carries on the principal resonance about the molecule are not universally applicable. Thus in the triphenylmethyl anion, $[(C_6H_5)_3C]^-$, the negative charge is expected to move from atom to atom in very nearly the same way as the positive charge in the triphenylcarbonium cation, $[(C_6H_5)_3C]^+$. It is observed that the spectra of the anion and the cation are similar, but not identical.

In an aromatic nitro compound such as picric acid resonance of the type



is expected, giving rise to the color of the substance. The aromatic hydrocarbons have large polarizabilities in the planes of their molecules, because

of the mobility of the p_z electrons. Hence an aromatic hydrocarbon molecule lying parallel to a molecule of picric acid or a similar substance at the minimum distance of approach (about 3.5 Å) would stabilize the structures with separated charges in the picric acid molecule by effectively increasing the dielectric constant in the expression for the Coulomb energy. (The orientation of permanent dipole molecules would not in general contribute to this effective dielectric constant, because of the high frequency of the charge-resonance; the effective orientation of the surrounding dipoles would be produced only by the average rather than the instantaneous dipole moment of the central molecule. Hence large electronic polarizability of surrounding molecules is needed to produce the effect under discussion.) This would deepen the color of the picric acid. In addition it gives rise to an interaction potential between the picric acid molecule and the hydrocarbon molecule corresponding to attraction and formation of a bond with energy of the order of magnitude of 2 kcal./mole, as observed for molecular compounds of this type. We conclude that of the many possible explanations which have been suggested for molecular compounds of this sort[8] the polarization theory in the form described above is the probably correct one.[9]

Detailed accounts of the application of the theory of color to specific problems, including the methods of formulation and solution of the equations giving the energy levels and wave functions, will be published elsewhere.

I am indebted to Dr. S. Weinbaum, Dr. B. Hicks, Dr. D. P. Stevenson, and Mr. P. A. Shaffer, Jr., for assistance with the calculations, and to Dr. G. Schwarzenbach and Dr. L. G. S. Brooker for encouragement in this attack on the problem of the color of dyes.

[1] This theory was presented at the Symposium on Liquids and Polyatomic Molecules at the University of Chicago on June 26, 1939.

[2] A. Baeyer, *Ann.*, **354**, 152 (1907).

[3] E. Q. Adams and L. Rosenstein, *J. Am. Chem. Soc.*, **36**, 1472 (1914).

[4] C. R. Bury, *Ibid.*, **57**, 2115 (1935).

[5] L. Pauling, *Organic Chemistry*, 1888–1890, H. Gilman, Editor; Wiley, **1938.**

[6] R. S. Mulliken, *J. Chem. Phys.*, **7**, 121, 364, 570 (1939).

[7] H. Mohler, H. Forster, and G. Schwarzenbach, *Helv. Chim. Acta*, **20**, 654 (1937).

[8] See N. V. Sidgwick, T. W. J. Taylor, and W. Baker, *The Organic Chemistry of Nitrogen*, 261–265, Oxford, 1937.

[9] This opinion has been expressed also, on the basis of a somewhat less detailed mechanism, by A. J. Stosick, Dissertation, California Institute of Technology, 1939.

PRIME POWER GROUPS DETERMINED BY THE NUMBER OF THEIR SUBGROUPS

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Communicated September 26, 1939

Every cyclic group of order p^m, p being an arbitrary prime number, contains exactly $m - 1$ proper subgroups, that is, subgroups which are different from the identity and the entire group. A non-cyclic group of order p^m contains more than $m - 1$ proper subgroups since it contains not only at least one subgroup whose order is an arbitrary divisor of the order of the group but it also contains more than one subgroup for at least one such divisor. While the cyclic group of order p^m contains a minimum number of subgroups among the groups of this order the abelian group of type 1^m contains a maximum number of such subgroups. This results from the fact that in all other groups of order p^m there are restrictions either as regards orders or commutability which are lacking in this abelian group, which was called the *principal group* of order p^m by J. A. De Séguier, *Groupes Abstraits*, page 13 (1904).

It will be proved in the present article that with a single exception a group of order p^m contains at least $(m - 1)(p + 1)$ proper subgroups if it contains more than $m - 1$ such subgroups. The single exception is furnished by the quaternion group, in which $m = 3$ and $p = 2$ but which contains only four proper subgroups. The abelian group of order p^m and of type $m - 1, 1$ contains exactly $(m - 1)(p + 1)$ proper subgroups for every pair of values of m and p. One of these subgroups of each of the possible orders is non-cyclic and the p remaining ones are cyclic whenever m exceeds 2. It is well known that when $p = 2$ and $m = 3$ there is no non-abelian group which contains exactly $(m - 1)(p + L)$ proper subgroups and that when $p = 2$ and $m = 4$ there are two such non-abelian groups, viz., the dicyclic group of order 16 and the group of this order obtained by adjoining to the cyclic group of order 8 an operator of order 2 which transforms each of the operators of this cyclic group into its fifth power. The latter of these groups is conformal with the abelian group which contains the same number of proper subgroups.

If an abelian group of order p^m is non-cyclic and not of type $m - 1, 1$ it contains more than $(m - 1)(p + 1)$ proper subgroups since each of its subgroups appears in a non-cyclic subgroup whose order is p times its own order and the number of the subgroups of index p in a group of order p^α is $(p^\lambda - 1)/(p - 1)$, where λ is the number of its independent generators.[1] For at least one of its subgroups this non-cyclic subgroup, which may be the entire group, can be so selected that it has more then two inde-

pendent generators unless the entire group has only two such generators. In the latter case the theorem in question is evident since there are then more than $p + 1$ cyclic subgroups of order p^2. It remains to prove that every non-abelian group G of order p^m contains at least $(m - 1)(p + 1)$ proper subgroups unless it is the quaternion group.

Since G is non-cyclic each of its subgroups appears in a non-cyclic subgroup whose order is p times its own order unless it is one of the four well-known non-abelian groups of order 2^m, $m > 3$, which contain a cyclic subgroup of index 2. It will be convenient to exclude these groups for the present in order to simplify the statements. Let H be a proper subgroup of order p^α contained in G and let H' be a non-cyclic subgroup of G which includes H and is of order $p^\alpha + 1$. It is known that the number of the subgroups of order p^α in H' is $(p^\lambda - 1)/(p - 1)$ if λ is the number of the operators in a set of independent generators of H'. As H' contains at least $p + 1$ proper subgroups of order p^α the group G also contains at least this number of proper subgroups of the arbitrary order p^α, $\alpha < m$. Since α is any positive integer less than m it has been proved that G contains at least $(m - 1)(p + 1)$ proper subgroups.

It has already been noted that among the groups of order 2^m which involve a cyclic subgroup of index 2 the quaternion group involves a smaller number of proper subgroups than $(m - 1)(p + 1)$ and that the dicyclic group of order 16 contains exactly nine proper subgroups and hence satisfies this formula. In all other cases when $m > 2$ and G is not the octic group there is one and only one non-abelian group of order p^m which contains exactly $(m - 1)(p + 1)$ proper subgroups. Hence it results that: *A non-cyclic group of order p^m, p being a prime number, contains at least $(m - 1)(p + 1)$ proper subgroups except when $p = 2$ and $m = 3$. In this special case the quaternion group exists and contains only four proper subgroups. In all other cases when $m > 2$, except when $p = 2$ and $m < 5$, there are two and only two non-cyclic groups of order p^m which separately contain as few as $(m - 1)(p + 1)$ proper subgroups. These two groups are conformal and only one of them is abelian.*

It is well known that there is one and only one non-cyclic group of order 2^m which contains only one subgroup of order 2, where m is an arbitrary positive integer which exceeds 2, and that in this case none of the subgroups of order 4 which contain this subgroup of order 2 is non-cyclic. While this case was excluded from the considerations noted above it is easy to verify that it also satisfies the conditions of the noted formula except in the cases which were explicitly excluded in the italicized theorem given at the close of the preceding paragraph. Hence this theorem applies also to the groups of order 2^m which contain a cyclic subgroup of index 2. It may be noted that the non-abelian groups which contain exactly $(m - 1)(p + 1)$ proper subgroups are those which contain separately p and only

p non-invariant subgroups, $m > 3$, which are relatively commutative and that a prime power group which contains more than $(m - 1)(p + 1)$ proper subgroups and has an order which exceeds 8 contains $(m - 1)(p + 1) + kp^2$ such subgroups.

An infinite system of abelian groups in all of which $k = 1$ according to this formula is furnished by the groups of order p^4 and of type 2^2 and this is composed of all the abelian groups of order p^m for which $k = 1$. An infinite system of non-abelian groups of the same order in which $k = 1$ for all values of the prime number p is furnished by the extension of the abelian group of type 2, 1 by an operator of order p^2 which is commutative with the operators of order p in this abelian group and transforms the operator of order p^2 therein into a power of themselves but has for its pth power an operator which is not generated by such an operator of order p^2. When p is odd a simpler infinite system of groups which satisfies the condition that the total number of its proper subgroups is expressed by the given formula when $k = 1$ is furnished by the non-abelian groups of order p^3 which include no operator of order p^2. This is the only non-abelian group of order p^m for which $k = 1$ and which involves no operator of order p^2. In the special case when $p = 2$ such a non-abelian group does not exist.

In the abelian group of order p^m and of type 1^m the number of the subgroups of order p is $(p^m - 1)/(p - 1)$, and the number of the subgroups which include an arbitrary subgroup of order p^α, $\alpha < m$, is equal to the number of the subgroups in its abelian group of order $p^{m-\alpha}$ and of type $1^{m-\alpha}$. Hence this number is

$$\frac{(p^m - 1)(p^m - p)\ldots(p^m - p^{\alpha-1})}{(p^\alpha - 1)(p^\alpha - p)\ldots(p^\alpha - p^{\alpha-1})} = \frac{(p^m - 1)(p^{m-1} - 1)\ldots(p^{m-\alpha+1} - 1)}{(p^\alpha - 1)(p^{\alpha-1} - 1)\ldots(p - 1)}.$$

It was noted above that when $p = 2$ the abelian group of type 1^m is completely determined by the number of its subgroups of order p but that this is not the case when $p > 2$. In this case the number of the subgroups of every order which exceeds p is larger in the abelian group of order p^m and of type 1^m than in any other group of this order and hence the number of the subgroups of each one of these orders determines the group. It is known that the number of the different non-abelian groups of order p^m which separately involve no operator whose order exceeds p increases with the value of m and has no upper limit.

It was noted above that if the number of the proper subgroups of an abelian group of order p^m is exactly $p^2 + (m - 1)(p + 1)$ it is of order p^4 and of type 2^2 and that if a non-abelian group of this order contains exactly $p^2 + (m - 1)(p + 1)$ proper subgroups and involves no operator of order p^2 it is of order p^3, and p is an odd prime number. It was also noted that for every prime number there is a non-abelian group which

contains exactly $p^2 + (m - 1)(p + 1)$ proper subgroups and is of order p^4. It is not difficult to prove that this is the only such group. Since the number of the proper subgroups of only one order in such a group is of the form $p^2 + p + 1$ and the number of the subgroups of every other order is $p + 1$ it results that the largest order of a cyclic subgroup contained in such a group is p^2 since otherwise there would be more than $p + 1$ subgroups in the group for more than one order. Hence the following theorem: *There is one and only one abelian group of order p^m which has exactly $p^2 + (m - 1)(p + 1)$ proper subgroups, viz., the abelian group of order p^4 and of type 2^2. When p is even there is also only one such non-abelian group but when p is odd there are two such non-abelian groups whose orders are p^3 and p^4, respectively. A cyclic group of an arbitrary order contains a smaller number of subgroups than any other abelian group of this order.*

[1] G. A. Miller, *Collected Works*, **1**, 320 (1935).

ON BASIS SYSTEMS FOR GROUPS OF IDEAL CLASSES IN A PROPERLY IRREGULAR CYCLOTOMIC FIELD

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In other articles the writer considered the irregular class group consisting of the qth powers of all ideals in the field $k(\zeta)$; $\zeta = e^{2i\pi/l}$; l an odd prime, where the class number of $k(\zeta)$ is H_1H_2, $H_2 = q_2$, with $(q_2, l) = 1$; $q = q_1q_2$ and $H_1 = l^h q_1$; $(l, q_1) = 1$. Here H_2 is the so-called second factor of the class number, and since it is prime to l, the field is called *properly irregular*. It was found that the number of invariants of this group is e, where e is the number of distinct Bernoulli numbers in the set, $B_1, B_2, \ldots B_{l_1}$; $l_1 = (l - 3)/2$, which are divisible by l; $B_a = (-1)^{a-1}b_{2a}$; $(b + 1)^n = b_n$, $n > 1$, and setting b_s for b^s in the expansion of the left member.

Let these distinct B's be

$$B_{a_1}, B_{a_2}, \ldots B_{a_e} \tag{1}$$

and let

$$\begin{gathered} l^{h_1}, l^{h_2}, \ldots l^{h_e}; \\ h = h_1 + h_2 + \ldots + h_e; \end{gathered} \tag{2}$$

be the invariants. Then a basis $C_1, C_2, \ldots, C_e$, may be selected for the group so that

$$C_i^{s-m_i} = 1;\ i = 1, 2, \ldots, e;\ m_i = r^{(l-2a_i)l^{h_i}}; \tag{3}$$

where r is a primitive root of l and s denotes the substitution (ζ/ζ^r). Also

$$B_{d_i} \equiv 0 (\text{mod } l^{h_i}),$$

where

$$d_i = \frac{l^{h_i}(2a_i - 1) + 1}{2},$$

but

$$B_{d_i} \not\equiv 0 (\text{mod } l^{h_i+1}),$$

where

$$d_i' = \frac{l^{h_i+1}(2a_i - 1) + 1}{2}.$$

In other papers[2] the writer examined the class number of the field $k(\zeta)$ for each value of $l < 619$ and found that each such field was properly irregular. In order to accomplish this we considered the unit

$$\left.\begin{aligned} E_n &= \epsilon^R; \\ R &= 1 + sr^{-2n} + s^2r^{-4n} + \ldots + s^{l_1}r^{-2nl_1} \\ \epsilon &= \left(\frac{(1-\zeta^r)(1-\zeta^{-r})}{(1-\zeta)(1-\zeta^{-1})}\right)^{1/2} \end{aligned}\right\} \tag{4}$$

and s is a symbolic power as before defined. In another paper[3] it was shown that the second factor of the class number of $k(\zeta)$ is prime to l if and only if none of the units $E_n;\ n = 1, 2, \ldots l_1$ is the lth power of a unit in $k(\zeta)$.

Also it is known that if E_n is an lth power then $B_n \equiv 0 (\text{mod } l)$. It was also shown[4] that, employing the notation of (1) that in each of the special fields mentioned above a prime ideal in $k(\zeta)$ may be found such that

$$\left(\frac{E_{a_i}}{\mathfrak{p}}\right) \neq 1;\ i = 1, 2, \ldots, e;$$

where the expression on the left is an lth power character in $k(\zeta)$. For example, in the case $l = 379, a_1 = 50, a_2 = 87, r = 2, \mathfrak{p}$ is a prime ideal divisor of the prime 4549.

The fact that this extensive induction showed that in all cases it was sufficient to use the same prime ideal for each of the values of a led me to examine the basis classes of the irregular class group more thoroughly in some respects than heretofore and yielded the theorem given at the end of this

paper. Induction has led to many results in the number theory of rational integers but the situation is rather unusual, apparently, in connection with the theory of algebraic numbers. Elsewhere[1] it was proved that

$$E_{a_1}, E_{a_2}, \ldots E_{a_e} \tag{5}$$

form a basis for the singular primary numbers of $k(\zeta)$ in the sense that if

$$E_{a_1}^{x_1} E_{a_2}^{x_2} \ldots E_{a_e}^{x_e} = \gamma^l; \ \gamma \text{ in } k(\zeta);$$

then $x_i \equiv 0 (\text{mod } l)$; $i = 1, 2, \ldots, e$.

In view of this and a theorem due to Kummer[5] an infinity of distinct ideal primes $\mathfrak{p}$ in $k(\zeta)$ may be found such that

$$\left(\frac{E_i}{\mathfrak{p}}\right) \neq 1; \ i = 1, 2, \ldots, e. \tag{5a}$$

If $\mathfrak{p}$ is one such prime, set

$$\mathfrak{p}_i = \sum_{c=0}^{l-2} (\mathfrak{p}(\zeta^{r^c}))^m \tag{6}$$

where

$$m = r^{c(2a_i-1)l^h};$$

$\mathfrak{p}(\zeta^d)$ denotes the ideal obtained from $\mathfrak{p}$ by the substitution (ζ/ζ^d) and l^h is the order of the irregular class group; it then follows easily that

$$\mathfrak{a}_i^{g_i} \sim 1, \tag{7}$$

where

$$g_i = s - r^{(l-2a_i)l^h},$$

and we are using symbolic powers as in (4). Let the classes to which the ideals $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_e$ belong be denoted, respectively, by $A_1, A_2, \ldots A_e$. We shall then show that these form a basis for the irregular class group of $k(\zeta)$.

Since the C's in (3) form a basis for this group, then

$$A_i = C_1^{k_1} C_2^{k_2} \ldots C_e^{k_e}.$$

If s denotes the substitution (ζ/ζ^r), write the above relation in additive form and apply the substitutions

$$s, s^2, \ldots, s^{l-2},$$

in turn, we find for $n = 0, 1, \ldots, l-2$, using (3) and (7),

$$r^{ni^h(l-2a_i)} A_i = \sum_{t=1}^{e} k_t r^{ni^h t(l-2a_t)} C_t. \tag{8}$$

Raise each side of this relation to the power $r^{-nl^h(l-2a_i)}$ and add the resulting relations for each n, we find that the coefficient of A_i is $(l-1)$ and the coefficient of C_i is

$$k_i \sum_n r^T,$$

where

$$T = ml^{h_i}(l-2a_i)(-l^{h-h_i}+1),$$

and since

$$r^{l^{h_i}(l-1)} \equiv 1 (\text{mod } l^{h_i})$$

and the order of C_i is l^{h_i}, then the coefficient of C_i reduces to $(l-1)k_i$. The coefficient of $C_j;\ j \neq i$, is

$$V = h_j \sum_n r^v,$$

where

$$v = -nl^h(l-2a_i) + nl^{h_j}(l-2a_j),$$

and if we set

$$w = (l-2a_j) - (l-2a_i)l^{h-h_i},$$

we find

$$V = \frac{r^{(l-1)l^{h_i}w} - 1}{r^{l^{h_i}w} - 1}. \tag{9}$$

Now

$$w \equiv 2a_i - 2a_j (\text{mod } l-1),$$

so that the denominator in (9) is $\not\equiv 0 \pmod{l}$, since $i \neq j$, and consequently

$$V \equiv 0 (\text{mod } l^{h_j}). \tag{10}$$

It will now be shown that $k_i \not\equiv 0 (\text{mod } l)$. For $k_i \equiv 0 (\text{mod } l)$ implies that, using (8) and (10),

$$(\theta)\mathfrak{a}_i = (\omega)\mathfrak{b}_i^l$$

for some ideal $\mathfrak{b}_i$, and hence, since E_i is singular primary,

$$\left(\frac{E_i}{\mathfrak{a}_i}\right) = \left(\frac{E_i}{\mathfrak{b}_i}\right)^l = 1. \tag{10a}$$

Let

$$E_a^{(N(\mathfrak{p})-1)/l} \equiv \zeta^t (\text{mod } \mathfrak{p}),$$

where $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$ in $k(\zeta)$ and since

$$N(\mathfrak{p}(\zeta^{r^c})) = N(\mathfrak{p}),$$

we obtain, using the substitution (ζ/ζ^{r^c}),

$$E_a(\zeta^{r^c})^{(N(\mathfrak{p})-1)/l} \equiv \zeta^{ir^c} (\mathrm{mod}\ \mathfrak{p}\ (\zeta^{r^c})),$$

and employing (4) we find

$$E_a(\zeta)^r = E_a(\zeta)^{r^{2a}\sigma^l},$$

and hence

$$E_a(\zeta)^{(N(\mathfrak{p})-1)/l \cdot r^{2ac}} \equiv \zeta^{ir^c} (\mathrm{mod}\ \mathfrak{p}\ (\zeta^{r^c})),$$

whence

$$\left(\frac{E_a(\zeta)}{\mathfrak{p}(\zeta^{r^c})}\right)^{r^{2ac}} = \left(\frac{E_a(\zeta)}{\mathfrak{p}}\right)^{r^c},$$

or

$$\left(\frac{E_a(\zeta)}{\mathfrak{p}(\zeta^{r^c})}\right)^{r^c(2a-1)} = \left(\frac{E_a(\zeta)}{\mathfrak{p}}\right),$$

and using (6), we have

$$\left(\frac{E_a(\zeta)}{\mathfrak{a}}\right) = \left(\frac{E_a(\zeta)}{\mathfrak{p}}\right)^{l-1},$$

and with (10a) this gives

$$\left(\frac{E_a(\zeta)}{\mathfrak{p}}\right) = 1,$$

which contradicts (5a). Hence k_i in (7) is prime to l, and (8), with what follows it, gives

$$(l-1)A_i = (l-1)k_iC_i,$$

and since $(l, l-1) = 1$, we obtain a result which may be put in the multiplicative form

$$A_i = C_i^{k_i}.$$

Now $(k_i, l) = 1$ and the C's form a basis for the class group so it then easily follows that

$$A_1, A_2, \ldots A_e$$

also form a basis for the irregular class group of $k(\zeta)$. Whence the

THEOREM. *There exists an infinity of ideal primes* $\mathfrak{p}$ *in the properly irregular cyclotomic field* $k(\zeta)$; $\zeta = e^{2i\pi/l}$; l *an irregular prime, such that if* l^h *is the number of classes in the irregular class group of* $k(\zeta)$, *and*

$$B_{a_1}, B_{a_2}, \ldots B_{a_e}$$

are the distinct Bernoulli numbers in the set

$$B_1, B_2, \ldots B_{(l-3)/2},$$

which are divisible by l, *then the ideals*

$$\mathfrak{a}_i = \sum_{c=0}^{l-2} (\mathfrak{p}(\zeta^{r^c}))^m,$$

with

$$m = r^{c(2a_i-1)l^h},$$

$i = 1, 2, \ldots, e$, *belong to a set of* e *distinct irregular classes in* $k(\zeta)$ *which form a basis for the irregular class group of* $k(\zeta)$.

From this we obtain immediately the

COROLLARY. *In any properly irregular cyclotomic field* $k(\zeta)$ *it is possible to find a class of ideals* C *such that any class in the irregular class group of* $k(\zeta)$ *may be expressed as a certain power product formed by* C *and its conjugate classes.*

In another paper[6] we obtained a result which obviously complements the theorem given above.

[1] These PROCEEDINGS, **15**, 202–207 (1929); "On the Composition of the Group of Ideal Classes in a Properly Irregular Cyclotomic Field," *Monatshefte für Mathematik und Physik*, 1939. Throughout these papers, where the relation $C_i^{s-r^l-2a_i} = 1$ is given, read $l^{h_i}(l - 2a_i)$ in lieu of $(l - 2a_i)$ in the exponent of r.

[2] *Duke Math. Journal*, **3**, 569–584 (1937); **5**, 418–427 (1939).

[3] These PROCEEDINGS, **16**, 743–749 (1930).

[4] *Duke Math. Journal*, **3**, 578 (1937); **5**, 418–419 (1939).

[5] Hilbert, *Bericht über die Theorie der Algebraischen Zahlkörper*, *Jahresbericht der Deutschen Math. Verein*, 426–428 (1897); *Werke*, 276–278.

[6] These PROCEEDINGS, **24**, 332 (1938).

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